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Trentelman, Hendrikus; Willems, Jan C.

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Guaranteed roll-off in a class of high-gain feedback design problems

Harry L. TRENTELMAN *
and Jan C. WILLEMS

Mathematics Institute, P.O. Box 800, 9700 AV Groningen, The Netherlands

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A number of synthesis problems associated with (almost) disturbance decoupling by state or measurement feedback is considered. Starting from a mathematical definition of the notion of high-frequency roll-off, known results on the solvability of these problems are generalized to the situation in which we require their solvability together with a certain guaranteed roll-off between disturbance and control. The conditions are formulated in terms of the solvability and approximate solvability of certain matrix equations in rational functions.

Keywords: High-gain feedback, Roll-off, Rational matrix equations, Almost invariant subspaces, Disturbance decoupling.

1. Introduction

Consider the finite-dimensional linear time-invariant system

$$\Sigma: \quad \begin{aligned} \dot{x} &= Ax + Bu + Gd, \\ y &= Cx, \quad z = Hx, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$, $y \in \mathbb{R}^p$ and $z \in \mathbb{R}^l$ are the state, control, disturbance, measurement and to-be-controlled output, respectively. The almost disturbance decoupling problem with measurement feedback, ADDPM, is the problem of finding a sequence of dynamic compensators

$$\Sigma_c(\varepsilon): \quad \begin{aligned} \dot{w} &= K_\varepsilon w + L_\varepsilon y, \\ u &= M_\varepsilon w + F_\varepsilon y, \end{aligned} \quad \varepsilon > 0, \tag{1.2}$$

such that the resulting sequence of closed-loop impulse-response matrices between d and z converges to zero in $L_1[0, \infty)$-norm. Necessary and sufficient conditions for the solvability of the above problem were obtained in [1]. These conditions were formulated in terms of the supremal L_1-almost controlled invariant subspace $\mathscr{V}_b^*$ 'contained in' ker H [2] and the infimal L_1-almost conditionally invariant subspace $\mathscr{S}_b^*$ 'containing' im G. Let Σ be represented by its transfer function matrix

$$\begin{pmatrix} z(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} d(s) \\ u(s) \end{pmatrix} \tag{1.3}$$

and $\Sigma_c(\varepsilon)$ by

$$u(s) = F_\varepsilon(s) y(s),$$

where $G_{ij}(s)$ and $F_\varepsilon(s)$ are related to (1.1) and (1.2) via

$$G_{11}(s) = H(Is - A)^{-1}G,$$

$$G_{12}(s) = H(Is - A)^{-1}B,$$

$$G_{21}(s) = C(Is - A)^{-1}G,$$

$$G_{22}(s) = C(Is - A)^{-1}B$$

and

$$F_\varepsilon(s) = F_\varepsilon + M_\varepsilon(Is - K_\varepsilon)^{-1}L_\varepsilon.$$

In the closed-loop system, the transfer matrix $G_\varepsilon(s)$ from d to z is given by the expression

$$G_\varepsilon(s) = G_{11}(s) + G_{12}(s)(I - F_\varepsilon(s)G_{22}(s))^{-1}F_\varepsilon(s)G_{21}(s). \tag{1.4}$$

Therefore, ADDPM amounts to finding a sequence $\{F_\varepsilon(s)\}_{\varepsilon>0}$ of proper rational $(m \times p)$ matrices such that

$$\|G_\varepsilon(s)\|_1 \to 0 \quad (\varepsilon \downarrow 0). \tag{1.5}$$

Here, for any rational matrix $R(s)$, $\|R(s)\|_1$ denotes the $L_1[0, \infty)$-norm of the inverse Laplace transform of $R(s)$. In [2] it is proven that (1.5) implies that the $L_p[0, \infty)$-induced norm in the closed-loop system from d to z can be made arbi-

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trarily small. In the closed-loop system the transfer matrix between the disturbance d and the control u is given by

$$T_\varepsilon(s)=(I-F_\varepsilon(s)G_{22}(s))^{-1}F_\varepsilon(s)G_{21}(s). \quad (1.6)$$

It can be shown that, if ADDPM is solvable, then the compensator sequence $\{F_\varepsilon(s)\}_{\varepsilon>0}$ can be constructed in such a way that as $\varepsilon\downarrow 0$, i.e. as the accuracy of decoupling between disturbance d and output z increases, the sequence $\{T_\varepsilon(s)\}_{\varepsilon>0}$ converges to a rational matrix $T(s)$. Of course, this limit $T(s)$ reflects the nature of the high-gain control action that is needed to achieve the desired asymptotic decoupling between d and z. In general, $T(s)$ may or may not be proper. It if is not proper, then this synthesis may be unacceptable because too much control action is needed to achieve the design purpose (i.e. almost disturbance decoupling). If $T(s)$ is proper but not strictly proper, then there may be a high-frequency feedthrough from the control to the disturbance. Consequently one may want to make sure that $T(s)$ has as large as possible an excess of poles over zeros. A large pole–zero excess increases the capability to attenuate signals which are of high frequency (high-frequency roll-off). Therefore it is of interest to know if a compensator sequence $\{F_\varepsilon(s)\}_\varepsilon>_0$ can be constructed such that almost disturbance decoupling is achieved, together with a certain prespecified guaranteed high-frequency roll-off. We formalize these ideas as follows:

Definition 1.1. Let $T(s)$ be a rational matrix. Then its *high-frequency roll-off* $r(T)$ will be defined by

$$r(T):=\max\left\{k\in\mathbb{Z}\mid \lim_{s\to\infty}s^kT(s)<\infty\right\}. \quad (1.7)$$

If $T=0$, we will define $r(T):=\infty$. Note that $r(T)\geqslant 0$ if and only if $T(s)$ is proper and $r(T)\geqslant 1$ if and only if $T(s)$ is strictly proper. Also note that for any $\rho\in\mathbb{Z}$, $r(T)\geqslant\rho$ if and only if the in the Laurent expansion of $T(s)$ the terms corresponding to the powers $s^{-\rho+k}$, $k=1,2,\ldots$, vanish identically.

A few words on the kind of convergence of the sequence of disturbance-to-control transfer matrices $T_\varepsilon(s)$ that will be considered in this paper are in order. Let $\mathscr{D}'_+$ be the space of matrix-valued distributions of given dimensions and with support in $[0,\infty)$. Let $\mathscr{S}'\subset\mathscr{D}'_+$ be the space of tempered distributions (see [3]). For any $\sigma_0\in\mathbb{R}$, let $\mathscr{S}'(\sigma_0)$ be the space of $T\in\mathscr{D}'_+$ such that

$$e^{-\sigma t}T\in\mathscr{S}',\quad \forall\sigma>\sigma_0.$$

A sequence $\{T_\varepsilon\}_{\varepsilon>0}$ converges in $\mathscr{S}'(\sigma_0)$ if $\{e^{-\sigma t}T_\varepsilon\}_{\varepsilon>0}$ converges in $\mathscr{S}'$ for all $\sigma>\sigma_0$. Such a sequence a fortiori converges in $\mathscr{D}'_+$. It is the latter kind of convergence we will consider for the transfer matrices $T_\varepsilon(s)$ between d and u:

Definition 1.2. We will say that $T_\varepsilon(s)$ converges in $\mathscr{S}'(\sigma_0)$ to $T(s)$ as $\varepsilon\downarrow 0$ if $L^{-1}T_\varepsilon$ converges to $L^{-1}T$ in the topology of $\mathscr{S}'(\sigma_0)$. Here L^{-1} denotes the inverse Laplace transform.

We will extensively use the following result on the convergence of distributions (cf. [3], Remark 1, p. 307).

Fact 1.3. *Let $\{T_\varepsilon(s)\}_{\varepsilon>0}$ be a sequence of rational matrices and $T(s)$ be a rational matrix. Suppose that $T_\varepsilon(s)-T(s)$ converges to zero as $\varepsilon\to 0$, uniformly on compact subsets of* Re $s>\sigma_0$. *Moreover, suppose that the Euclidean norms $\|T_\varepsilon(s)+T(s)\|$ are all bounded from above in* Re $s>\sigma_0$ *by a polynomial in $|s|$, independent of ε.*

Then $T_\varepsilon(s)$ converges in $\mathscr{S}'(\sigma_0)$ to $T(s)$ as $\varepsilon\downarrow 0$ (in the sense of Definition 1.2).

In the present paper, the main problem we want to consider is the following:

Definition 1.4. Let $\rho\in\mathbb{Z}$ be arbitrary but fixed. We will say that $(\mathrm{ADDPM})_\rho$, the *almost disturbance decoupling problem with measurement feedback and guaranteed roll-off* ρ, is solvable if and only if there exists a sequence of compensators $\{\Sigma_c(\varepsilon)\}_{\varepsilon>0}$ such that (1.5) holds and such that, for some $\sigma_0\in\mathbb{R}$,

$$T_\varepsilon(s)\to T(s)\quad \text{in}\ \mathscr{S}'(\sigma_0)\ \text{as}\ \varepsilon\downarrow 0,$$

where $T(s)$ is a rational matrix with $r(T)\geqslant\rho$.

2. Exact disturbance decoupling with positive roll-off by state feedback

In this section we will consider the system

$$\dot{x}=Ax+Bu+Gd,\quad z=Hx.$$

For any state feedback $F:\mathbb{R}^n\to\mathbb{R}^m$, denote

$$T_F(s) := F(Is - A_F)^{-1}G,$$

where $A_F := A + BF$. We define:

Definition 2.1. Let $\rho \in \mathbb{Z}$ be positive. We will say that $(\mathrm{DDP})_\rho$, the *disturbance decoupling problem with guaranteed roll-off* ρ, is solvable if and only if there exists F such that

$$H(Is - A_F)^{-1}G = 0 \quad \text{and} \quad r(T_F) \geqslant \rho.$$

Note that if one takes $\rho = 1$ in the above definition, one obtains the ordinary disturbance decoupling problem, DDP (see [4]). In the following let

$$\langle A_F \,|\, \mathrm{im}\, G\rangle := \sum_{i=1}^{n} A_F^{i-1}\, \mathrm{im}\, G.$$

For any subspace $\mathscr{K} \subset \mathbb{R}^n$ and positive $\rho \in \mathbb{Z}$, define

$$\mathscr{R}^{\rho-1}(\mathscr{K}) := \bigcap_{i=1}^{\rho} A^{-i+1}\mathscr{K}. \tag{2.1}$$

Here

$$A^{-i+1}\mathscr{K} := \left\{ x \in \mathbb{R}^n \,|\, A^{i-1}x \in \mathscr{K} \right\}.$$

Note that the subspaces $\mathscr{R}^i(\mathscr{K})$ are nested according to

$$\mathscr{K} = \mathscr{R}^0(\mathscr{K}) \supset \mathscr{R}^1(\mathscr{K}) \supset \cdots.$$

Let $\mathscr{V}^*$ denote the supremal (A, B)-invariant subspace contained in ker H (see [4]). The following theorem states that a necessary and sufficient condition for $(\mathrm{DDP})_\rho$ to be solvable is that the disturbances enter $\mathscr{V}^*$ sufficiently 'deeply'.

Theorem 2.2. *Let $\rho \in \mathbb{Z}$ be positive. Then $(DDP)_\rho$ is solvable if and only if*

$$\mathrm{im}\, G \subset \mathscr{R}^{\rho-1}(\mathscr{V}^*).$$

Proof. ($\Leftarrow$) Let $x_1, \ldots, x_k$ be a basis for $\mathscr{R}^{\rho-1}(\mathscr{V}^*)$. For $j \in \boldsymbol{k}$ and $i = 0, \ldots, \rho - 1$ there holds

$$A^i x_j \in \mathscr{R}^{\rho-i-1}(\mathscr{V}^*) \subset \mathscr{V}^*.$$

Define

$$\mathscr{W} := \mathrm{span}\left\{ A^i x_j \,|\, j \in \boldsymbol{k},\, i = 0, \ldots, \rho - 1 \right\}.$$

Let $\mathscr{L}$ be such that $\mathscr{W} \oplus \mathscr{L} = \mathscr{V}^*$. Choose F_0 such that $A_{F_0}\mathscr{V}^* \subset \mathscr{V}^*$. Define $F: \mathbb{R}^n \to \mathbb{R}^m$ by $F \,|\, \mathscr{W} := 0$, $F \,|\, \mathscr{L} := F_0 \,|\, \mathscr{L}$ and F arbitrary outside $\mathscr{V}^*$. Then clearly $A_F \mathscr{V}^* \subset \mathscr{V}^*$. Finally, since $F \,|\, \mathscr{W} = 0$ and $A^i \mathscr{R}^{\rho-1}(\mathscr{V}^*) \subset \mathscr{W}$, we have

$$FA^iG = 0, \quad i = 0, \ldots, \rho - 2.$$

($\Rightarrow$) From $\langle A_F \,|\mathrm{im}\, G\rangle \subset \ker H$ we obtain that

$$\langle A_F \,|\, A_V^i\, \mathrm{im}\, G\rangle \subset \ker H, \quad \forall i.$$

Since

$$FA_F^iG = 0, \quad i = 0, \ldots, \rho - 2,$$

there follows

$$\langle A_F \,|\, A^i\, \mathrm{im}\, G\rangle \subset \ker H$$

and hence

$$A^i\, \mathrm{im}\, G \subset \mathscr{V}^*, \quad i = 0, \ldots, \rho - 1. \qquad \square$$

For our purposes it is convenient to state the above results in terms of the solvability of a certain rational matrix equation. Let $\mathbb{R}^{m \times q}(s)$ be the space of $(m \times q)$ matrices with entries in $\mathbb{R}(s)$, the field of real rational functions. For $\rho \in \mathbb{Z}$, let

$$\mathbb{R}_\rho^{m \times q}(s) \subset \mathbb{R}^{m \times q}(s)$$

be the space of rational matrices $T(s)$ with the property that $r(T) \geqslant \rho$. Consider the following linear equation:

$$\text{(L)} \quad G_{12}(s)X(s) + G_{11}(s) = 0. \tag{2.2}$$

Here, $G_{12}(s)$ and $G_{11}(s)$ are as defined in Section 1. The following result then states that the solvability of $(\mathrm{DDP})_\rho$ is equivalent to the solvability of (L) over the space $\mathbb{R}_\rho^{m \times q}(s)$:

Theorem 2.3. *Let $\rho \in \mathbb{Z}$ be positive. Then* (L) *is solvable over $\mathbb{R}_\rho^{m \times q}(s)$ if and only if*

$$\mathrm{im}\, G \subset \mathscr{R}^{\rho-1}(\mathscr{V}^*). \qquad \square$$

3. Almost disturbance decoupling with non-positive roll-off by state feedback

Again, consider the system

$$\dot{x} = Ax + Bu + Gd, \quad z = Hx.$$

It is well known (see [2]) that if DDP is not solvable, then it may be possible to approximately decouple d and z up to any degree of accuracy. In this section we will consider a generalization of this problem, the almost disturbance decoupling problem ADDP (see [2]), while incorporating a certain guaranteed upper bound on the number of differentiators that are asymptotically needed to realize the decoupling transfer matrix between the disturbance d and the control u. In our terminology, for $\rho \in \mathbb{Z}$, $\rho \leqslant 0$, a guaranteed upper bound $-\rho$ to the number of differentiators needed to realize $T(s)$ (i.e. the highest non-negative power of s in the Laurent expansion of $T(s)$) corresponds to a guaranteed roll-off $r(T) \geqslant \rho$. Recall that for the state feedback $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^m$,

$$T_{F_\varepsilon}(s) := F_\varepsilon(Is - A_{F_\varepsilon})^{-1}G.$$

Definition 3.1, Let $\rho \in \mathbb{Z}$ be non-positive. We will say that $(\text{ADDP})_\rho$, the *almost disturbance decoupling problem with guaranteed roll-off* ρ, is solvable if and only if there exists a sequence of state feedback matrices $\{F_\varepsilon\}_{\varepsilon>0}$ and $T(s) \in \mathbb{R}_\rho^{m\times q}(s)$ such that

$$\|H(Is - A_{F_\varepsilon})^{-1}G\|_1 \to 0$$

and

$$T_{F_\varepsilon}(s) \to T(s) \quad \text{in } \mathscr{S}'(0)$$

as $\varepsilon \downarrow 0$.

Consider now the almost controllability subspace algorithm (ACSA)′ [2], which computes recursively the following subspaces:

$$\begin{aligned} \mathscr{S}_{\ker H}^{k+1} &:= \operatorname{im} B + A(\mathscr{S}_{\ker H}^{k} \cap \ker H), \\ \mathscr{S}_{\ker H}^{0} &:= \{0\}. \end{aligned} \tag{3.1}$$

This recursion defines a non-decreasing sequence of subspaces which reaches a limit after a finite number of steps. Moreover, this limit equals $\mathscr{S}_{\ker H}^{n} = \mathscr{R}_b^*$, the supremal L_1-almost controllability subspace 'contained in' ker H (see [2]). Since $\mathscr{V}_b^* = \mathscr{V}^* + \mathscr{R}_b^*$, the main result of this section states that a necessary and sufficient condition for (ADDP) to be solvable is that the disturbances enter $\mathscr{V}_b^*$ sufficiently 'deeply'.

Theorem 3.2. *Let* $\rho \in \mathbb{Z}$ *be non-positive. Then* $(ADDP)_\rho$ *is solvable if and only if*

$$\operatorname{im} G \subset \mathscr{V}^* + \mathscr{S}_{\ker H}^{-\rho+1}.$$

Proof (outline). ($\Rightarrow$) Let $\{F_\varepsilon\}_{\varepsilon>0}$ be as in Definition 3.1. Note that

$$H(Is - A_{F_\varepsilon})^{-1}G = H(Is - A)^{-1}(G + BT_{F_\varepsilon}(s)).$$

It can be shown that the left-hand side converges to 0 pointwise in Re $s > 0$ and that $T_{F_\varepsilon}(s) \to T(s)$ pointwise in Re $s > 0$. Hence we obtain

$$0 = H(Is - A)^{-1}(G + BT(s))$$

in Re $s > 0$, and, consequently, for all $s \in \mathbb{C}$. Since $T(s) \in \mathbb{R}_\rho^{q\times m}(s)$, the conclusion then follows from [5].

($\Leftarrow$) (i) Decompose

$$\mathscr{V}^* + \mathscr{S}_{\ker H}^{-\rho+1} = \mathscr{V}^* \oplus \mathscr{L},$$

where $\mathscr{L}$ is the sum of singly generated almost controllability subspaces [6]. Using a result analogous to [6], Lemma 7.5, let $\mathscr{X} \subset \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ be such that

$$\langle A \mid \operatorname{im} B\rangle = \mathscr{X} \oplus \mathscr{V}^* \oplus \mathscr{L},$$

$$A_F\mathscr{V}^* \subset \mathscr{V}^*, \quad A_F(\mathscr{X} \oplus \mathscr{V}^*) \subset \mathscr{X} \oplus \mathscr{V}^*$$

and

$$\operatorname{Re} \sigma(A_F|(\mathscr{X} \oplus \mathscr{V}^*)/\mathscr{V}^*) < 0.$$

(ii) Let $\mathscr{L}_\varepsilon$ be a sequence of (A, B)-invariant subspaces converging to $\mathscr{L}$ and let $\tilde{F}_\varepsilon : \mathscr{L}_\varepsilon \to \mathbb{R}^m$ be such that

$$(A + B\tilde{F}_\varepsilon)\mathscr{L}_\varepsilon \subset \mathscr{L}_\varepsilon$$

and

$$\sigma(A + B\tilde{F}_\varepsilon|\mathscr{L}_\varepsilon) = \left\{-\frac{1}{\varepsilon}, \ldots, -\frac{1}{\varepsilon}\right\}$$

(cf. the construction in [7]). For ε sufficiently small,

$$\langle A \mid \operatorname{im} B\rangle = \mathscr{X} \oplus \mathscr{V}^* \oplus \mathscr{L}_\varepsilon.$$

Define now

$$F_\varepsilon | \mathscr{L}_\varepsilon := \tilde{F}_\varepsilon | \mathscr{L}_\varepsilon, \quad F_\varepsilon|(\mathscr{X} \oplus \mathscr{V}^*) := F|(\mathscr{X} \oplus \mathscr{V}^*)$$

and $F_\varepsilon := 0$ outside $\langle A \mid \operatorname{im} B\rangle$.

(iii) It can then be proven that

$$\|H(Is - A_{F_\varepsilon})^{-1}G\|_1 \to 0 \quad (\varepsilon \downarrow 0),$$

that $T(s) \in \mathbb{R}_\rho^{m\times q}(s)$ exists such that

$$T_{F_\varepsilon}(s) - T(s) \to 0 \quad (\varepsilon \downarrow 0)$$

uniformly on compact sets in $\operatorname{Re} s > 0$ and that the (Euclidean) norms $\|T_{F_\epsilon}(s) - T(s)\|$ are bounded from above in $\operatorname{Re} s > 0$ by a polynomial in $|s|$, independent of ε. The results then follows from Fact 1.3. □

We will now formulate the above result again in terms of the solvability of the rational matrix equation (L) as given by (2.2). Recall that $\mathbb{R}_1^{m\times q}(s)$ is the space of all strictly proper real rational $(m\times q)$ matrices.

Definition 3.3. Let $\rho \in \mathbb{Z}$ be non-positive. We will say that (L) is approximately ρ-solvable over $\mathbb{R}_1^{m\times q}(s)$ if and only if there is a sequence

$$\{X_\varepsilon(s)\}_{\varepsilon>0} \subset \mathbb{R}_1^{m\times q}(s)$$

and $X(s) \in \mathbb{R}_\rho^{m\times q}(s)$ such that

$$\|G_{12}(s)X_\varepsilon(s) + G_{11}(s)\|_1 \to 0$$

and

$$X_\varepsilon(s) \to X(s) \quad \text{in } \mathscr{S}'(0)$$

as $\varepsilon \downarrow 0$.

The following theorem now follows easily (see also [1], Prop. A.2):

Theorem 3.4. *Let $\rho \in \mathbb{Z}$ be non-positive. Then the following statements are equivalent*:

(i) (L) *is approximately ρ-solvable over* $\mathbb{R}_1^{m\times q}(s)$.

(ii) (L) *is solvable over* $\mathbb{R}_\rho^{m\times q}(s)$.

(iii) $\operatorname{im} G \subset \mathscr{V}^* + \mathscr{S}_{\ker H}^{-\rho+1}$. □

4. Almost disturbance decoupling with guaranteed roll-off by measurement feedback

We will now consider the main problem of this paper. It turns out that necessary and sufficient conditions for the solvability of $(\mathrm{ADDPM})_\rho$ can be obtained for $\rho > 0$ $(\rho \leqslant 0)$ by requiring the solvability of both $(\mathrm{DDP})_\rho$ $((\mathrm{ADDP})_\rho)$ *and* the solvability of the almost disturbance decoupled estimation problem, ADDEP (see [1]). The latter problem requires the existence of an observer, having the measurement y as its input and an estimate $\hat{z}$ of z as its output, such that the L_1-norm of the impulse response from d to the estimation error $e := z - \hat{z}$ is arbitrarily small. Hence, our main result generalizes the separation principle as stated in [1], p. 1079:

Theorem 4.1. (i) (positive guaranteed roll-off). *Let $\rho \in \mathbb{Z}$ be positive. Then $(ADDPM)_\rho$ is solvable if and only if*

$$\operatorname{im} G \subset \mathscr{R}^{\rho-1}(\mathscr{V}^*) \quad \textit{and} \quad \mathscr{S}_b^* \subset \ker H. \tag{4.1}$$

(ii) (non-positive guaranteed roll-off). *Let $\rho \in \mathbb{Z}$ be non-positive. Then $(ADDPM)_\rho$ is solvable if and only if*

$$\operatorname{im} G \subset \mathscr{V}^* + \mathscr{S}_{\ker H}^{-\rho+1} \quad \textit{and} \quad \mathscr{S}_b^* \subset \ker H. \tag{4.2}$$

The proof of this theorem will be given through a series of lemmas involving the solvability and approximate solvability of the following linear rational matrix equation (cf. [1]). Consider

$$(\mathrm{L}') \quad G_{12}(s)X(s)G_{21}(s) + G_{11}(s) = 0. \tag{4.3}$$

Recall that $\mathbb{R}_0^{m\times p}(s)$ is the space of all proper real rational $(m\times p)$ matrices.

Definition 4.2. Let $\rho \in \mathbb{Z}$. We will say that (L′) is ρ-solvable over $\mathbb{R}^{m\times p}(s)$, if and only if there is $X(s) \in \mathbb{R}^{m\times p}(s)$ such that

$$X(s)G_{21}(s) \in \mathbb{R}_\rho^{m\times q}(s).$$

We will say that (L′) is approximately ρ-solvable over $\mathbb{R}_0^{m\times p}(s)$ if and only if there exists a sequence

$$\{X_\varepsilon(s)\}_{\varepsilon>0} \subset \mathbb{R}_0^{m\times p}(s),$$

$T(s) \in \mathbb{R}_\rho^{m\times q}(s)$ and a real σ_0 such that as $\varepsilon \to 0$

$$\|G_{11}(s) + G_{12}(s)X_\varepsilon(s)G_{21}(s)\|_1 \to 0 \tag{4.4}$$

and

$$X_\varepsilon(s)G_{21}(s) \to T(s) \quad \text{in } \mathscr{S}'(\sigma_0). \tag{4.5}$$

Make the following important observation (cf. [1], Lemma 4.2):

Lemma 4.3. *Let $\rho \in \mathbb{Z}$. Then $(ADDPM)_\rho$ is solvable if and only if* (L′) *is approximately ρ-solvable over* $\mathbb{R}_0^{m\times p}(s)$.

Proof. $(\Rightarrow)$ There is a sequence

$$\{F_\varepsilon(s)\}_{\varepsilon>0} \subset \mathbb{R}_0^{m\times p}(s)$$

and $T(s) \in \mathbb{R}_\rho^{m\times q}(s)$ such that

$$\|G_\varepsilon(s)\|_1 \to 0$$

and

$$T_\varepsilon(s) \to T(s) \quad \text{in } \mathscr{S}'(\sigma_0)$$

as $\varepsilon \downarrow 0$, where $G_\varepsilon(s)$ and $T_\varepsilon(s)$ are given by (1.4) and (1.6) respectively. Now take

$$X_\varepsilon(s) := (I - F_\varepsilon(s) G_{22}(s))^{-1} F_\varepsilon(s).$$

Then $X_\varepsilon(s) \in \mathbb{R}_0^{m\times p}(s)$ and (4.4) and (4.5) hold.

($\Leftarrow$) Let $\{X_\varepsilon(s)\}_{\varepsilon>0} \subset \mathbb{R}_0^{m\times p}(s)$ satisfy (4.4) and (4.5). Take

$$F_\varepsilon(s) := X_\varepsilon(s)(I + G_{22}(s) X_\varepsilon(s))^{-1}.$$

Then $F_\varepsilon(s) \in \mathbb{R}_0^{m\times p}(s)$ and, since

$$X_\varepsilon(s) = (I - F_\varepsilon(s) G_{22}(s))^{-1} F_\varepsilon(s),$$

it follows that (1.4) is satisfied and that $T_\varepsilon(s)$, as defined by (1.6), satisfies

$$T_\varepsilon(s) \to T(s) \quad \text{in } \mathscr{S}'(\sigma_0). \qquad \square$$

The following result states that approximate ρ-solvability over the space of proper rational matrices is equivalent to exact ρ-solvability over the space of all rational matrices. This nice result will enable us to continue the proof of Theorem 4.1 in a purely algebraic way.

Lemma 4.4. *Let $\rho \in \mathscr{Z}$. (L′) is approximately ρ-solvable over $\mathbb{R}_0^{m\times p}(s)$ if and only if (L′) is ρ-solvable over $\mathbb{R}^{m\times p}(s)$.*

Proof. ($\Leftarrow$) For suitable rational matrices

$$A(s) \in \mathbb{R}^{ql\times pm}(s), \quad b(s) \in \mathbb{R}^{ql}(s)$$

and

$$C(s) \in \mathbb{R}^{qm\times pm},$$

(L′) can be written as

$$A(s)x(s) + b(s) = 0,$$

whereas the constraint

$$X(s) G_{21}(s) \in \mathbb{R}_\rho^{m\times q}(s)$$

can be translated into

$$C(s)x(s) \in \mathbb{R}_\rho^{mq}(s).$$

Note that $A(s)x(s) + b(s) = 0$ is a special case of (L) as given by (2.2) (see also [1], Comment 1). Therefore it follows from the proof of Theorem 3.2 that a sequence

$$\{x_\varepsilon(s)\}_{\varepsilon>0} \subset \mathbb{R}_1^{pm}(s)$$

exists and $\hat{x}(s) \in \mathbb{R}^{pm}(s)$ such that the following hold:

(1) $\|A(s)x_\varepsilon(s) + b(s)\|_1 \to 0$,

(2) $x_\varepsilon(s) - \hat{x}(s) \to 0$ uniformly on compact sets in $\operatorname{Re} s > 0$, and

(3) the norms $\|x_\varepsilon(s) - x(s)\|$ are bounded from above in $\operatorname{Re} s > 0$ by a polynomial in $|s|$, independent of ε.

Consider $\mathbb{R}^{pm}(s)$ as a linear space over the field $\mathbb{R}(s)$. Obviously $A(s)$ defines a linear map on $\mathbb{R}^{pm}(s)$. Denote $\mathscr{A} := \ker A(s)$ and let $\mathscr{X} \subset \mathbb{R}^{pm}(s)$ be such that $\mathscr{A} \oplus \mathscr{X} = \mathbb{R}^{pm}(s)$. Let $(0 \vdots A_2(s))$ be a matrix for $A(s)$ in this decomposition. Note that $A_2(s)$ is left-invertible. Decompose

$$x(s) = \left(x_1(s)^{\mathrm{T}} \vdots x_2(s)^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$x_\varepsilon(s) = \left(x_{\varepsilon,1}(s)^{\mathrm{T}} \vdots x_{\varepsilon,2}(s)^{\mathrm{T}}\right)^{\mathrm{T}}$$

and

$$\hat{x}(s) = \left(\hat{x}_1(s)^{\mathrm{T}} \vdots \hat{x}_2(s)^{\mathrm{T}}\right)^{\mathrm{T}}.$$

From (1) it follows that

$$A(s)x_\varepsilon(s) + b(s) = A_2(s)x_{\varepsilon,2}(s) + b(s) \to 0 \quad (\varepsilon \downarrow 0),$$

pointwise in $\operatorname{Re} s > 0$. From (2) it follows that

$$A_2(s)x_{\varepsilon,2}(s) + b(s) \to A_2(s)\hat{x}_2(s) + b(s) \quad (\varepsilon \downarrow 0),$$

pointwise in $\operatorname{Re} s > 0$. Hence we obtain

$$A_2(s)\hat{x}_2(s) = -b(s) = A_2(s)x_2(s),$$

from which it follows that $\hat{x}_2(s) = x_2(s)$. Write

$$x_1(s) = x_1^+(s) + x_1^-(s),$$

where $x_1^+(s)$ is strictly proper and $x_1^-(s)$ is a polynomial vector. Let k be a positive integer such that for all $\varepsilon > 0$

$$(\varepsilon s + 1)^{-k} x_1^-(s)$$

is strictly proper. Denote

$$x_{\varepsilon,1}^*(s) := x_1^+(s) + (\varepsilon s + 1)^{-k} x_1^-(s).$$

Clearly $x_{\varepsilon,1}^*(s)$ is strictly proper. Define now

$$\tilde{x}_\varepsilon(s) := \left(x^*_{\varepsilon,1}(s)^{\mathrm{T}} \vdots\, x_{\varepsilon,2}(s)^{\mathrm{T}}\right)^{\mathrm{T}}.$$

It follows immediately that

(1) $\|A(s)\tilde{x}_\varepsilon(s) + b(s)\|_1 \to 0$ ($\varepsilon \downarrow 0$),

(2) $\tilde{x}_\varepsilon(s) - x(s) \to 0$ ($\varepsilon \downarrow 0$), uniformly on compact sets in $\operatorname{Re} s > 0$, and

(3) the norms $\|\tilde{x}_\varepsilon(s) - x(s)\|$ are bounded from above in $\operatorname{Re} s > 0$ by a polynomial in $|s|$, independent of ε.

Finally, it can then be seen that

$$C(s)\tilde{x}_\varepsilon(s) \to C(s)x(s) \quad \text{in } \mathscr{S}'(\sigma_0),$$

for some σ_0 depending on the rational matrix $C(s)$.

($\Rightarrow$) From (4.4) and (4.5) we obtain

$$\begin{pmatrix} A(s) \\ C(s) \end{pmatrix} x_\varepsilon(s) + \begin{pmatrix} b(s) \\ t(s) \end{pmatrix} \to 0$$

($\varepsilon \downarrow 0$) pointwise in $\operatorname{Re} s > \max\{0, \sigma_0\}$, for some $t(s) \in \mathbb{R}^{mq}_\rho(s)$. It can then be proven that a vector $x(s) \in \mathbb{R}^{pm}(s)$ exists, such that

$$\begin{pmatrix} A(s) \\ C(s) \end{pmatrix} x(s) + \begin{pmatrix} b(s) \\ t(s) \end{pmatrix} = 0. \qquad \square$$

As the last major step in our proof of Theorem 4.1, we will show that the ρ-solvability of (L′) is equivalent to both the solvability over $\mathbb{R}^{m\times q}_\rho(s)$ of the equation (L) as given by (2.2) and the solvability over $\mathbb{R}^{l\times p}(s)$ of the linear equation

$$\text{(L″)} \quad X(s)G_{21}(s) + G_{11}(s). \tag{4.6}$$

For this we need a rather special result on the existence of a normal form for rational matrices. In the following, a permutation matrix will be a $q \times q$ square matrix P, obtained by interchanging the columns of the $q \times q$ identity matrix I_q arbitrarily.

Lemma 4.5. *Let $R(s)$ be a real rational $(q \times r)$ matrix $(q \geq r)$ of full rank r. Then there exist a $(q \times q)$ permutation matrix P and a bijective rational $(r \times r)$ matrix $\Lambda(s)$ such that*

$$R(s) = PQ(s)\Lambda(s),$$

where $Q(s)$ is a rational $(q \times r)$ matrix of the form

$$Q(s) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & & \vdots \\ \vdots & * & \ddots & 0 \\ \vdots & & \ddots & 1 \\ \vdots & & & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \end{pmatrix} \tag{4.7}$$

with zeros above the diagonal, ones on the diagonal and proper rational functions below the diagonal.

Proof. Omitted.

For any linear space $\mathscr{X}$, decomposed as $\mathscr{X} = \mathscr{R} \oplus \mathscr{S}$, the projection $\pi : \mathscr{X} \to \mathscr{X}$ of $\mathscr{X}$ onto $\mathscr{S}$ along $\mathscr{R}$ will be defined by

$$\pi \mid \mathscr{S} = 1 \mid \mathscr{S} \quad \text{and} \quad \pi \mid \mathscr{R} = 0 \mid \mathscr{R}.$$

In the following, let E_0 denote the $q \times (q-r)$ matrix defined by

$$E_0 := \begin{pmatrix} 0 \\ \cdots \\ I_{q-r} \end{pmatrix}.$$

Note that if $Q(s)$ has the form (4.7), then the composite matrix $(Q(s) \vdots E_0)$ is bicausal, i.e. it is proper and has a proper inverse. This property will be crucial in the proof of the following:

Lemma 4.6. *Consider the linear space $\mathbb{R}^q(s)$. Let $\mathscr{V} \subset \mathbb{R}^q(s)$ be a subspace. Then there exists a subspace $\mathscr{E} \subset \mathbb{R}^q(s)$ such that $\mathscr{V} \oplus \mathscr{E} = \mathbb{R}^q(s)$ and such that the projection π of $\mathbb{R}^q(s)$ onto $\mathscr{E}$ along $\mathscr{V}$ has the property that*

$$\pi \mathbb{R}^q_0(s) \subset \mathbb{R}^q_0(s).$$

Proof. Suppose that $\dim \mathscr{V} = r$ and let $R(s)$ be a full-rank rational matrix such that $\mathscr{V} = \operatorname{im} R(s)$. Factorize

$$R(s) = PQ(s)\Lambda(s),$$

where $Q(s)$ has the canonical form (4.7). Define a subspace $\mathscr{E}_0 \subset \mathbb{R}^q(s)$ by $\mathscr{E}_0 := \operatorname{im} E_0$ and define $\mathscr{E} := P\mathscr{E}_0$. Note that $\mathscr{V} = P \operatorname{im} Q(s)$ and that $\mathbb{R}^q(s) = \operatorname{im} Q(s) \oplus \mathscr{E}_0$. It follows that $\mathbb{R}^q(s) = \mathscr{V} \oplus \mathscr{E}$.

Moreover, it can be seen that a matrix of π with respect to the natural basis of $\mathbb{R}^q(s)$ is given by

$$\pi(s) := (0_{q\times r} \vdots PE_0)(PQ(s) \vdots PE_0)^{-1}.$$

Since

$$(PQ(s) \vdots PE_0)^{-1} = (Q(s) \vdots E_0)^{-1}P^{-1},$$

we find that $\pi(s)$ is proper or, equivalently, that the subset $\mathbb{R}_0^q(s)$ is invariant under π. □

We can now prove the following important lemma:

Lemma 4.7. *Let $\rho \in \mathbb{Z}$. Then* (L′) *is ρ-solvable over $\mathbb{R}^{m\times p}(s)$ if and only if* (L) *is solvable over $\mathbb{R}_\rho^{m\times q}(s)$ and* (L″) *is solvable over $\mathbb{R}^{l\times p}(s)$.*

Proof. ($\Rightarrow$) is a triviality.

($\Leftarrow$) Let

$$X_1(s) \in \mathbb{R}_\rho^{m\times q}(s) \quad \text{and} \quad X_2(s) \in \mathbb{R}^{l\times p}(s)$$

be solutions to (L) and (L″) respectively. Apply the foregoing lemma with $\mathscr{V} = \ker G_{21}(s)$ and define

$$X_1'(s) := X_1(s)\pi(s).$$

Note that

$$\ker G_{21}(s) \subset \ker G_{11}(s).$$

It follows that $X_1'(s)$ is a solution to (L), since

$$G_{12}(s)X_1'(s) \mid \ker G_{21}(s) = 0 = -G_{11}(s) \mid \ker G_{21}(s)$$

and

$$G_{12}(s)X_1'(s) \mid \mathscr{E} = G_{12}(s)X_1(s) \mid \mathscr{E} = -G_{11}(s) \mid \mathscr{E}.$$

Also note that $X_1'(s) \in \mathbb{R}_\rho^{m\times q}(s)$. Finally, since

$$\ker G_{21}(s) \subset \ker X_1'(s),$$

$X_1'(s)$ can be written as

$$X_1'(s) = X(s)G_{21}(s)$$

for some $X(s) \in \mathbb{R}^{m\times p}(s)$. □

We will now complete the proof of Theorem 4.1.

Proof of Theorem 4.1. For $\rho \in \mathbb{Z}$, $\rho \geqslant 0$ ($\rho \leqslant 0$) the solvability of (L) over $\mathbb{R}_\rho^{m\times q}(s)$ is equivalent to

$$\operatorname{im} G \subset \mathbb{R}^{\rho-1}(\mathscr{V}^*)$$

$$(\operatorname{im} G \subset \mathscr{V}^* + \mathscr{S}_{\ker H}^{-\rho+1})$$

and the solvability of (L″) over $\mathbb{R}^{l\times p}(s)$ is equivalent to $\mathscr{S}_b^* \subset \operatorname{im} G$ (see [1], Prop. 2). □

Note that the proof of Theorem 4.1 required some more analysis than the analogous result in [1] on the solvability of ADDPM without roll-off constraints. The solvability of the latter problem can be formulated purely in terms of the solvability of a linear equation over an arbitrary field (see [1], App. B).

There are two important special cases of Theorem 4.1 that we want to state in a separate corollary:

Corollary 4.8. (i) *$(ADDPM)_1$ is solvable if and only if*

$$\operatorname{im} G \subset \mathscr{V}^* \quad \textit{and} \quad \mathscr{S}_b^* \subset \ker H.$$

(ii) *$(ADDPM)_0$ is solvable if and only if*

$$\operatorname{im} G \subset \mathscr{V}^* + \operatorname{im} B \quad \textit{and} \quad \mathscr{S}^* \subset \ker H. \quad □$$

Note that the first result states that the conditions for solvability of ADDPM with disturbance-to-control transfer matrices having a *strictly proper* limit, are that both the disturbance decoupling problem by state feedback, DDP (see [4], Ch. 4), and the almost disturbance decoupling estimation problem, ADDEP (see [1]), are solvable. If in the above statement we change strictly proper by *proper*, then the conditions become equivalent to the solvability of both $(\text{ADDP})_0$ and ADDEP.

5. Extensions

Remark 5.1. The results of Section 2 can be dualized to obtain conditions for the solvability of the disturbance decoupling estimation problem [1,8] with guaranteed (positive) roll-off ρ, $(\text{DDEP})_\rho$. This problem is concerned with the system

$$\dot{x} = Ax + Gd, \quad y = Cx, \quad z = Hz.$$

It is said to be solvable if an observer

$$\dot{\omega} = K\omega + Ly, \quad \hat{z} = M\omega,$$

exists such that the transfer matrix $H(s)$ between d

and $e := z - \hat{z}$ vanishes identically and such that the observer transfer matrix

$$M(Is - K)^{-1}L$$

is in $\mathbb{R}_{\rho}^{l \times p}(s)$. Let $\mathscr{S}^*$ be the infimal (C, A)-invariant subspace containing im G [8], and for $\rho \in \mathbb{Z}$, $\rho > 0$ define

$$\mathscr{T}^{\rho-1}(\mathscr{S}^*) := \sum_{i=1}^{\rho} A^{i-1}\mathscr{S}^*.$$

It can then be proven that $(\text{DDEP})_\rho$ is solvable if and only if

$$\mathscr{T}^{\rho-1}(\mathscr{S}^*) \subset \ker H. \tag{5.1}$$

Moreover, this condition can be shown to be equivalent to the solvability over $\mathbb{R}_{\rho}^{l \times p}(s)$ of the linear equation

$$G_{11}(s) + X(s)G_{21}(s) = 0.$$

Remark 5.2. The results of Section 3 can be dualized to obtain conditions for the solvability of the almost disturbance decoupled estimation problem [1] with guaranteed (non-positive) roll-off ρ, $(\text{AD-DEP})_\rho$. This problem will be said to be solvable if, for the observed system as above, a sequence of observers $\{\Sigma_{\text{obs}}(\varepsilon)\}_{\varepsilon>0}$,

$$\Sigma_{\text{obs}}(\varepsilon): \quad \dot{\omega} = K_\varepsilon \omega + L_\varepsilon y, \quad \hat{z} = M_\varepsilon \omega,$$

exists such that the sequence of transfer matrices $H_\varepsilon(s)$ between the disturbance d and the estimation error $e := \hat{z} - z$ satisfies

$$\|H_\varepsilon(s)\|_1 \to 0 \quad (\varepsilon \downarrow 0)$$

and such that, for some $W(s) \in \mathbb{R}_{\rho}^{l \times p}(s)$,

$$M_\varepsilon(Is - K_\varepsilon)^{-1}L_\varepsilon \to W(s) \quad \text{in } \mathscr{S}'(0)$$

as $\varepsilon \downarrow 0$. Following [1], define an almost complementary observability subspace algorithm (ACOSA)′ by

$$\begin{aligned} \mathscr{V}_{\text{im}G}^{k+1} &= \ker C \cap A^{-1}\left(\mathscr{V}_{\text{im}G}^{k} + \text{im } G\right), \\ \mathscr{V}_{\text{im}G}^{0} &= \mathbb{R}^n. \end{aligned} \tag{5.2}$$

It can then be shown that for $\rho \in \mathbb{Z}$, $\rho \leqslant 0$, $(\text{AD-DEP})_\rho$ is solvable if and only if

$$\mathscr{S}^* \cap \mathscr{V}_{\text{im}G}^{-\rho+1} \subset \ker H. \tag{5.3}$$

This condition can be shown to be equivalent to the approximate ρ-solvability over $\mathbb{R}_{0}^{l \times p}(s)$ of the equation

$$G_{11}(s) + X(s)G_{21}(s) = 0.$$

Finally we would like to mention the connection between the PID-disturbance decoupled estimation problem ([1], Def. 9) and condition (5.3). It can in fact be proven that if (5.3) is satisfied, then $-\rho$ gives an upper bound to the order of differentiation in the PID-observer needed to achieve disturbance-decoupled estimation.

Remark 5.3. The final remark we want to make is that we could also dualize the results of Section 4. We would then find that for $\rho > 0$ ($\rho \leqslant 0$) the conditions im $G \subset \mathscr{V}_b^*$ and (5.1) (resp. (5.3)) taken together are necessary and sufficient for the approximate ρ-solvability over $\mathbb{R}_{0}^{m \times p}(s)$ of (L′), now in the sense that $G_{12}(s)X_\varepsilon(s)$ converges to an element in $\mathbb{R}_{0}^{l \times p}(s)$.

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